

NORMAL SUBGROUPS OF THE ALGEBRAIC FUNDAMENTAL GROUP OF AFFINE CURVES IN POSITIVE CHARACTERISTIC

AMÍLCAR PACHECO, KATHERINE F. STEVENSON, AND PAVEL ZALESSKII

ABSTRACT. Let $\pi_1(C)$ be the algebraic fundamental group of a smooth connected affine curve, defined over an algebraically closed field of characteristic $p > 0$ of countable cardinality. Let N be a normal (resp. characteristic) subgroup of $\pi_1(C)$. Under the hypothesis that the quotient $\pi_1(C)/N$ admits an infinitely generated Sylow p -subgroup, we prove that N is indeed isomorphic to a normal (resp. characteristic) subgroup of a free profinite group of countable cardinality. As a consequence, every proper open subgroup of N is a free profinite group of countable cardinality.

1. INTRODUCTION

Let C be a smooth connected affine curve defined over an algebraically closed field k of characteristic $p > 0$. Let $\pi_1(C)$ be its algebraic fundamental group. From the proof of Abhyankar's Conjecture, it is known which finite groups are quotients of $\pi_1(C)$ (HARBATER [Ha94, Theorem 6.2] and RAYNAUD [Ra94, corollaire 2.2.2]). However, not much else is known about the actual structure of the subgroups of $\pi_1(C)$. The objective of the paper is to give a description of the normal subgroup structure of the fundamental group $\pi_1(C)$. We study the infinite index normal subgroup structure of $\pi_1(C)$ and prove that the majority of it coincides with the infinite index normal subgroup structure of a free profinite group.

Theorem 1.1. *Let $\pi_1(C)$ be the algebraic fundamental group of a smooth connected affine curve C defined over an algebraically closed field k of characteristic $p > 0$ of countable cardinality. Let N be a normal (resp. characteristic) subgroup of $\pi_1(C)$ such that a Sylow p -subgroup M_p of the quotient group $M = \pi_1(C)/N$ is infinitely generated. Then N is isomorphic to a normal (resp. characteristic) subgroup of a free profinite group of countable rank.*

This gives an almost complete description of such subgroups if one takes into account the known structure of the normal subgroups of a free profinite group (cf. [RiZa00, chapter 8]). For example one deduces immediately from Theorem 1.1 the following result.

Corollary 1.2. *Every proper open subgroup of N is a free profinite group of countable rank.*

In the proof of the above theorem we use a strong result due to POP [Po95, Theorem B] which asserts that every finite *quasi p -embedding problem* of $\pi_1(C)$

Date: June 23, 2008.

The first and third authors were partially supported by CNPq research grants 305731/2006-8 and 307823/2006-7, respectively. They were also supported by Edital Universal CNPq 471431/2006-0.

is properly solvable (*cf.* also [Ha03, Corollary 4.6]). This result also implies ABHYANKAR's conjecture mentioned above. In fact, the latter result provides the existence of a proper solution for a specific finite quasi- p embedding problem.

Unfortunately, ABHYANKAR's conjecture does not hold for smooth connected projective curves defined over algebraically closed fields of characteristic $p > 0$. Indeed, in the case of a finite group that is an extension of a finite group of order prime to p by a finite p -group, the two first authors established in [PaSt00, Theorem 1.3] a necessary and sufficient condition (*cf.* [PaSt00, condition A]) for ABHYANKAR's conjecture to hold for such a group. Subsequently this was extended by BORNE to the case where the first group does not necessarily have order prime to p , using modular representation theory (*cf.* [Bo04, Theorem 1.1]).

We also use in the proof of the above theorem the fact that ARTIN-SCHREIER covers of the projective line can be obtained with arbitrarily high genus. These covers are necessarily wildly ramified at only one point. Even more, there is no finite étale non-trivial cover of the projective line. So, the method used in the proof gives no information about the validity of a similar result in the case of smooth connected projective curves defined over algebraically closed fields of positive characteristic. It remains an open problem whether some other strategy might lead to such a result. However, there is one reason to be skeptical *a priori*. In contrast with patching techniques for affine curves, patching techniques for projective curves do not in general preserve the base curve.

In fact, the following result was proved in [St96, Theorem 3.1]. Let G be a finite group and suppose G is generated by two subgroups G_1 and G_2 . Assume that for $i = 1, 2$ the group G_i is a GALOIS group over a smooth connected projective curve C_i , étale except at one point $x_i \in C_i$. Moreover assume that the ramification is tame over each x_i with cyclic inertia groups generated by $\tau_i \in G_i \subset G$ such that $\tau_1 = \tau_2^{-1}$ in G . Then G is realized as an étale GALOIS group over almost all smooth connected projective curves of genus $g_1 + g_2$ where g_i the genus of C_i . So, not only is the base curve not preserved, but also the genus of the curve obtained via patching techniques grows with respect to those of the original curves.

Nevertheless, results similar to Theorem 1.1, Corollary 1.2 and Theorem 2.9 were proved for the algebraic fundamental group of smooth connected projective curves defined over algebraically closed fields of characteristic zero by the third author in [Za05, Theorem 1.1, Corollary 2.5, Theorem 2.2], respectively. Consider then the following three objects:

- free profinite groups;
- algebraic fundamental groups of connected projective curves defined over algebraically closed fields of characteristic zero; and
- algebraic fundamental groups of connected affine curves defined over algebraically closed fields of countable cardinality and positive characteristic $p > 0$;

The results mentioned above lead us to think that although these three objects are dissimilar in many ways, when we consider the family of projective normal subgroups N of each Π such that the quotient Π/N has an infinitely generated Sylow p -subgroup, this family behaves in exactly the same way in the three cases. Therefore we say that their "*p-deep structure*" are alike.

In Section 2 we discuss some preliminaries on profinite group theory, and in Section 3 we discuss a particular family of finite p -groups related to the group N .

In Section 4 we solve the embedding problem which suffices to prove Theorem 2.9. It should be noted that we do not need any further patching result other than those already present in [Ku06]. We acknowledge the author of the preprint as having constructed an original solution in the case of the commutator subgroup of $\pi_1(C)$. One of our contributions was to observe that the *gestalt* of his construction did not require any additional information on the GALOIS groups of the covers. So we were able to replace finite *abelian covers* by finite *p-covers*. It should be noted however that our Corollary 1.2 implies the main result of KUMAR [Ku06, Theorem 4.1] (*cf.* Corollary 5.7) and is, in fact, more general than the latter cited result. Additionally, in his thesis [Ku07, §7.3 and Theorem 7.13] KUMAR comments that his result should hold if one replaces the commutator subgroup by kernel of an epimorphism of $\pi_1(C)$ to an infinitely generated abelian pro- p group. However, this result is also a consequence of Theorem 2.9, and we also show in Example 5.9 that Theorem 2.9 addresses situations which are not covered by the latter comment in [Ku07]. Finally, in Section 5 we obtain group theoretic consequences from Theorems 2.7 and 2.9, and in the Appendix we rewrite the proof of [Za05, Theorem 2.2] connecting it with the present paper (*cf.* Theorem 2.9).

2. PROFINITE GROUP THEORY

For the reader's convenience we review in this section MELNIKOV's characterization of accessible, normal, and characteristic subgroups of free profinite groups. We restrict the discussion to *second countable* profinite groups (*i.e.*, groups that have a countable basis of open subsets), since this is sufficient for our purposes and simplifies the terminology. Beyond this review, the main function of this section is to translate the problem of proving that each normal subgroup N of $\pi_1(C)$ is isomorphic to a normal subgroup of a free profinite group of countable rank into one of solving finite split embedding problems for N that arise from those for $\pi_1(C)$ (see Definition 2.4). This is done in two steps. First using a theorem of MELNIKOV (Theorem 2.7 below) we reduce our question to the problem to showing that given finite groups Γ and G the embedding problem \mathcal{E} defined by

$$(2.0.1) \quad \begin{array}{ccc} & N & \\ & \downarrow \psi & \\ \Gamma & \xrightarrow{\alpha} & G. \end{array}$$

admits a proper solution under certain additional hypotheses. Then Lemma 2.11 allows us, roughly speaking, to assume that \mathcal{E} is a split embedding problem.

2.1. Profinite groups.

Definition 2.2. A closed subgroup H of a profinite group Π is said to be *accessible*, if there exists a chain of closed subgroups of Π

$$(2.2.1) \quad H = G_\mu \leq \cdots \leq G_\lambda \leq \cdots \leq G_2 \leq G_1 = \Pi,$$

indexed by the ordinals smaller than a certain ordinal μ , such that

- $G_{\lambda+1} \triangleleft G_\lambda$ for all ordinals $\lambda \leq \mu$, and
- if ν is a limit ordinal such that $\nu \leq \mu$, then $G_\nu = \bigcap_{\lambda < \nu} G_\lambda$.

Remark 2.3. Recall that a subgroup K of a group Π is called *subnormal* if K is a member of a normal series of Π . Observe that H is accessible if and only if the image of H is subnormal in every finite quotient of Π .

Definition 2.4. Let G_1, G_2 and G_3 be profinite groups. An *embedding problem* \mathcal{E} consists of a pair of epimorphisms

$$(2.4.1) \quad \begin{array}{ccc} & & G_1 \\ & & \downarrow f_1 \\ G_2 & \xrightarrow{f_2} & G_3. \end{array}$$

The embedding problem \mathcal{E} is *properly solvable*, if there exists an epimorphism $f_3 : G_1 \rightarrow G_2$ such that $f_2 \circ f_3 = f_1$, i.e., the diagram (2.4.1) commutes. The map f_3 is called a *proper solution* for \mathcal{E} .

Definition 2.5. For a group G , let $M(G)$ denote the intersection of maximal normal subgroups of G and let $R_p(G)$ denote the kernel of the epimorphism to the maximal pro- p quotient.

If S is a finite simple group, $M_S(G)$ will denote the kernel of the epimorphism to the maximal direct power of S ; if S is a finite simple p -group, then S is necessarily isomorphic to the cyclic group of order p . Therefore we shall use the notation $M_p(G)$.

Definition 2.6. A second countable profinite group Π is said to be *homogeneous*, if given any finite groups Γ and G and an embedding problem \mathcal{E} defined by

$$(2.6.1) \quad \begin{array}{ccc} & & \Pi \\ & & \downarrow f \\ \Gamma & \xrightarrow{\alpha} & G, \end{array}$$

\mathcal{E} admits a proper solution under the additional hypotheses that $H := \text{Ker}(\alpha)$ is minimal normal and $H \leq M(\Gamma)$.

The next theorem collects facts originally proved by MELNIKOV about homogeneous profinite groups and it also can be found in [RiZa00, chapter 8].

Theorem 2.7. *Let Π be a profinite group. Then*

- (1) Π is homogeneous if and only if it is isomorphic to an accessible subgroup of free profinite group of countable rank;
- (2) Π is isomorphic to a normal subgroup of a free profinite group of countable rank if and only if Π is homogeneous and $\Pi/M_p(\Pi)$ is either trivial or infinite for every p ;
- (3) Π is isomorphic to a characteristic subgroup of a free profinite group if and only if Π is homogeneous and $\Pi/M_S(\Pi)$ is either trivial or infinite for every finite simple group S .
- (4) Π is free profinite of countable rank if and only if it is homogeneous and $\Pi/M_S(\Pi)$ is infinite for every finite simple group S .

We shall also need the following

Lemma 2.8. *Let P be a pro- p group and \mathcal{G} a finite quotient of P such that its minimal number of generators $d(\mathcal{G})$ is less than the minimal number of generators $d(P)$ of P . Then $\mathcal{G} \times \mathbb{Z}/p\mathbb{Z}$ is a quotient of P .*

Proof. Let t be a generator of P which is in the kernel K of the natural epimorphism of P onto \mathcal{G} and $\eta : P \rightarrow \mathbb{Z}/p\mathbb{Z}$ an epimorphism onto a group $\mathbb{Z}/p\mathbb{Z}$ of order p such that $\eta(t) \neq 1$. Then $P/(K \cap \text{Ker}(\eta)) \cong \mathcal{G} \times \mathbb{Z}/p\mathbb{Z}$ as needed. \square

2.8.1. Our key result is the following theorem.

Theorem 2.9. *Let $\pi_1(C)$ be the algebraic fundamental group of a smooth connected affine curve defined over an algebraically closed field of characteristic $p > 0$ of countable cardinality. Let N be a normal subgroup of $\pi_1(C)$ such that a Sylow- p subgroup M_p of the quotient group $M = \pi_1(C)/N$ is infinitely generated. Then N is homogeneous.*

2.10. Changing the embedding problem. It follows from Theorem 2.7 (item (1)) that in order to prove Theorem 2.9 it suffices to show that given finite groups Γ and G the embedding problem \mathcal{E} defined by

$$(2.10.1) \quad \begin{array}{ccc} & & N \\ & & \downarrow \psi \\ \Gamma & \xrightarrow{\alpha} & G. \end{array}$$

admits a proper solution under the additional hypotheses that $H := \text{Ker}(\alpha)$ is minimal normal and $H \leq M(\Gamma)$.

The next lemma is an extract of Lemma 2.4 and Example 2.5 (a) in [HaSt05].

Lemma 2.11. *Given a projective profinite group Π and a finite embedding problem*

$$(2.11.1) \quad \begin{array}{ccc} & & \Pi \\ & & \downarrow \psi \\ \Gamma & \xrightarrow{\alpha} & G. \end{array}$$

There exists a finite split embedding problem

$$(2.11.2) \quad \begin{array}{ccc} & & \Pi \\ & & \downarrow \psi' \\ \Gamma' & \xrightarrow{\alpha'} & G'. \end{array}$$

such that any proper solution of (2.11.2) induces a proper solution of (2.11.1). Moreover, if the kernel H of α is of order prime to p , then so is the kernel H' of α' .

Proof. Since Π is projective there is a weak solution $\psi' : \Pi \rightarrow \Gamma$ of (2.11.1). Let $G' := \psi'(\Pi)$. Let $\Gamma' := \Gamma \times_G G'$ be the pullback. Then we have the following commutative diagram

$$(2.11.3) \quad \begin{array}{ccc} & & \Pi \\ & & \downarrow \psi' \\ \Gamma' & \xrightarrow{\alpha'} & G' \\ \downarrow & & \downarrow \psi \\ \Gamma & \xrightarrow{\alpha} & G. \end{array}$$

Since Γ' is a subgroup of a direct product $\Gamma \times G'$, the diagram (2.11.3) yields the following split embedding problem

$$(2.11.4) \quad \begin{array}{ccc} & & \Pi \\ & & \downarrow \psi' \\ \Gamma' & \xrightarrow{\alpha'} & G' \end{array}$$

Given a proper solution $\lambda' : \Pi \twoheadrightarrow \Gamma'$ of (2.11.4), it is automatic that $\lambda := q \circ \lambda'$ is a proper solution of (2.10.1), where $q : \Gamma' \twoheadrightarrow \Gamma$ is the projection to the first coordinate.

The last statement follows from the fact that it is always possible to factor α' through the quotient of H' by the normal closure of a Sylow p -subgroup of H' . \square

Remark 2.12. It follows from Lemma 2.11 that in order to properly solve the embedding problem (2.10.1), it suffices to properly solve the associated finite split embedding problem

$$(2.12.1) \quad \begin{array}{ccc} & & N \\ & & \downarrow \psi' \\ \Gamma' & \xrightarrow{\alpha'} & G' \end{array}$$

(Warning: H' is not a minimal normal subgroup of Γ').

Remark 2.13. By [RiZa00, Lemma 8.3.8] there exists an open normal subgroup R of $\pi_1(C)$ containing N and an epimorphism $\varphi : R \twoheadrightarrow G$ such that $\varphi|_N = \psi$. Note that every R is the fundamental group of a finite étale cover C' of C . So after replacing $\pi_1(C)$ by $R = \pi_1(C')$, if necessary, we may assume the existence of the following commutative diagram:

$$(2.13.1) \quad \begin{array}{ccc} N & \xrightarrow{\quad} & \pi_1(C) \\ \downarrow \psi & \swarrow \varphi & \\ \Gamma & \xrightarrow{\alpha} & G \end{array}$$

It follows from (2.13.1) that for every subgroup $N \leq L \leq \pi_1(C)$, the restriction $\varphi|_L$ of φ to L is an epimorphism.

3. A FAMILY OF p -GROUPS

Terminology. For any ring R , let $\text{frac}(R)$ be its total ring of fractions. If R is a domain and $R \subset S$ is an extension of rings, then we say that S is *generically separable* as an R -algebra if $\text{frac}(S)$ is a separable $\text{frac}(R)$ -algebra and if no non-zero element of R becomes a zero-divisor of S . For any finite group G , a G -GALOIS R -algebra consists of a finite separable R -algebra S together with a group homomorphism $\rho : G \rightarrow \text{Aut}_R(S)$ with respect to which G acts simply transitively on the generic geometric fiber of $\text{Spec}(S) \rightarrow \text{Spec}(R)$. We say that a morphism of schemes $\phi : Y \rightarrow X$ is *generically separable* (resp. G -GALOIS with respect to a homomorphism $\rho : G \rightarrow \text{Aut}_X(Y)$) if X can be covered by affine open subsets $U = \text{Spec}(R)$ such that the ring extension $R \subset \mathcal{O}(\phi^{-1}(U))$ have the corresponding property. A morphism $\phi : Y \rightarrow X$ which is finite and generically separable (resp. finite and G -GALOIS) will be called a *cover* (resp. G -GALOIS cover).

3.1. A geometric translation. The results of this section reduce our problem to a problem in algebraic geometry. Specifically, given a G -GALOIS cover of a curve C arising from an *a priori* specified epimorphism $\varphi : \pi_1(C) \twoheadrightarrow G$ such that $\varphi|_N$ is still an epimorphism, we want to find a Γ -GALOIS cover of C that dominates the given G -cover. This Γ -GALOIS cover will correspond to an epimorphism $\lambda : \pi_1(C) \twoheadrightarrow \Gamma$ and we must show that $\lambda|_N$ is an epimorphism.

One of the crucial points in the construction of the aforementioned Γ -GALOIS cover is to produce finite étale covers of C which admit as subcovers finite étale GALOIS covers whose GALOIS groups are p -groups. These finite p -groups will not be arbitrary, in fact they are chosen as finite quotients of a SYLOW- p subgroup M_p of $M := \pi_1(C)/N$. In this section we focus on these groups by constructing the covers they determine. This is done in the language of function fields (see diagram (3.1.1)). For a complete description of our global strategy, see Subsection 3.4.

Let \overline{C} be a smooth compactification of C and $S := \overline{C} \setminus C$. Let $K := k(C)$ be the function field of C and $K_{\text{ur},S}$ the maximal GALOIS extension of K (inside some algebraic closure of K) which is unramified over the points of C . The fundamental group $\pi_1(C)$ can also be viewed as the GALOIS group $\text{Gal}(K_{\text{ur},S}/K)$.

Notation. For any smooth connected curve C defined over k , we will denote by $k(C)$ its function field. Given a profinite group Π we will denote by $d(\Pi)$ the minimal number of generators of Π .

We start by observing that since N is a closed subgroup of $\pi_1(C)$, by infinite GALOIS theory there exists a unique subextension K_M/K of $K_{\text{ur},S}/K$ such that K_M is field fixed by N . Moreover, since N is normal in $\pi_1(C)$, then K_M/K is GALOIS with $\text{Gal}(K_M/K) = M$. Similarly, since M_p is closed in M , denote by K_p the subfield of K_M fixed by M_p .

The SYLOW- p subgroup M_p of M can be described via the following projective limit $M_p = \varprojlim \Upsilon$, where Υ runs through the open subgroups of M containing M_p . Observe that since $M = \text{Gal}(K_M/K)$, each Υ is equal to $\text{Gal}(K_M/k(U_\Upsilon))$, where U_Υ is a finite étale cover of C . Alternatively, $M_p = \varprojlim \Upsilon^{(p)}$, where $\Upsilon^{(p)}$ denotes the maximal pro- p quotient of Υ . Let $K_{U,p}/k(U_\Upsilon)$ be the subextension of $K_M/k(U_\Upsilon)$ such that $\text{Gal}(K_{U,p}/k(U_\Upsilon)) = \Upsilon^{(p)}$. Furthermore, any finite quotient \mathcal{G} of M_p is indeed a finite quotient of some $\Upsilon^{(p)}$. Hence, \mathcal{G} will be equal to $\text{Gal}(k(U_p)/k(U_\Upsilon))$ for some finite étale GALOIS p -cover U_p of U_Υ . The following diagram summarizes

4. SOLVING THE EMBEDDING PROBLEM (2.10.1)

We have already observed in Section 3 that in order to obtain a proper solution for the embedding problem (2.10.1), it suffices to produce a finite étale GALOIS p -cover $T' \rightarrow U_\Upsilon$ such that $\text{Gal}(k(T')/k(U_\Upsilon))$ is a finite quotient of M_p and that the embedding problem in (3.4.1) defined by the pair of maps (α', φ'_1) admits a proper solution. Thus, in Subsection 4.4 we use patching techniques to build a curve T over $k[[t]]$ for which we can solve (α', φ'_1) . In Subsection 4.7.1 we specialize this solution to obtain a solution over k . Then in Subsection 4.10 we go on to use a group theoretic argument to solve (2.10.1).

Hypothesis 4.1. From this point through Subsection 4.10 our goal is to solve the embedding problem (2.10.1). It follows from Lemma 2.11 that it suffices to solve split embedding problem (2.12.1). So we suppose we are already in the latter case. We also assume that H is of order prime to p . As already mentioned above this implies that the kernel H' of the latter embedding problem also has order prime to p .

Remark 4.2. We actually do not need the hypothesis that H has order prime to p for Subsection 4.10. Moreover the case complementary to it is a consequence of the fact that every finite quasi p -embedding problem of the algebraic fundamental group of a smooth connected affine curve C defined over an algebraically closed field of characteristic $p > 0$ is properly solvable (cf. [Ha03, Corollary 4.6] and [Po95, Theorem B]). The argument is given in Subsection 4.11.

Remark 4.3. The main idea of our construction is the following. We start on one side with a G' -étale GALOIS cover of one the curves U_p described above. Recall that these curves satisfy the property that there exists an étale GALOIS cover U_p/U_Υ whose GALOIS group equals a finite quotient of M_p . On the other side we consider a H' -étale GALOIS cover of an ARTIN-SCHREIER cover of \mathbb{P}_k^1 . Remember that our goal is to solve the split embedding problem (2.12.1) and that H' is exactly the kernel of this embedding problem. After deforming these covers they are patched so as to produce a Γ' -GALOIS cover W of a curve T over $\text{Spec}(k[[t]])$. Taking the generic fiber of W and then specializing it, we will obtain a Γ' -GALOIS cover of a curve T' defined over k and this T' will be one of the U_p 's previously defined.

4.4. Solving the embedding problem for some T .

4.4.1. *Description of the G' -cover.* Let \mathcal{G} be a finite quotient of M_p and let $\phi_{U_p} : U_p \rightarrow U_\Upsilon$ be the finite étale GALOIS p -cover of U_Υ such that $\text{Gal}(U_p/U_\Upsilon) = \mathcal{G}$, where U_Υ is a finite étale cover of C satisfying $\text{Gal}(K_M/k(U_\Upsilon)) = \Upsilon$ and Υ is an open subgroup of M containing M_p (cf. diagram (3.1.1)). The inclusion of fundamental groups (3.3.1) and the argument presented in subsection 3.4 show us that there exists an epimorphism $\varphi_p : \pi_1(U_p) \twoheadrightarrow G'$ such that $\varphi_p = \varphi|_{\pi_1(U_p)}$ (cf. (2.13.1)). This means that there exists an étale GALOIS cover $V \rightarrow U_p$ with $G' = \text{Gal}(V/U_p)$.

Let \overline{U}_Υ be a smooth compactification of U_Υ . Let X be the normalization of \overline{U}_Υ in $k(U_p)$ and $\phi_X : X \rightarrow \overline{U}_\Upsilon$ the normalization morphism. The morphism ϕ_X extends ϕ_{U_p} to X , hence it is \mathcal{G} -GALOIS and étale over U_Υ . Similarly, let W_X be the normalization of X in $k(V)$ and $\phi_{W_X} : W_X \rightarrow X$ the normalization morphism. This is a G' -GALOIS cover étale over $\phi_X^{-1}(U_\Upsilon)$.

4.4.2. *Description of the H' -cover.* Let $\phi_Y : Y \rightarrow \mathbb{P}_y^1$ be an ARTIN-SCHREIER cover totally ramified over ∞ . Denote its equation by $z^p - z = f(y)$, where $f \in k[z]$ has degree ν relatively prime to p . The genus of Y can be arbitrarily high, since it is equal to $(p-1)(\nu-1)/2$. For a discussion of other cases of wildly ramified covers with arbitrarily large genus *cf.* [Pr06]. Thus, we may suppose that the genus $g(Y)$ of Y is at least $\max(2, r)$, where r denotes the minimal number of generators of H' . Since H' has order prime to p , it follows from [SGA1, Exp. X, Corollaire 3.10] that there exists an étale H' -GALOIS cover $\phi_{W_Y} : W_Y \rightarrow Y$. We reparametrize \mathbb{P}_y^1 so that Y is totally ramified over $y = 0$. Denote by $s \in Y$ the point lying over $y = 0$.

4.4.3. *Descending to \mathbb{P}^1 .* Since k is algebraically closed, then $k(U_\Upsilon)/k$ has a separating variable. Moreover, a stronger version of NOETHER's normalization theorem [Ei95, Corollary 16.18] implies that there exists a finite proper k -morphism $\phi_{U_\Upsilon} : U_\Upsilon \rightarrow \mathbb{A}_x^1$ which is furthermore generically separable. The branch locus of the morphism will be of codimension 1, so it will be étale away from finitely many points. After a reparametrization, if necessary, we may assume that none of these points lie over $x = 0$. Moreover, ϕ_{U_Υ} extends to a finite proper morphism $\phi_{\overline{U}_\Upsilon} : \overline{U}_\Upsilon \rightarrow \mathbb{P}_x^1$. Note that $\phi_{\overline{U}_\Upsilon}^{-1}(\{x = \infty\}) = \overline{U}_\Upsilon \setminus U_\Upsilon =: \mathcal{U}_\Upsilon$. Composing the latter morphism with the \mathcal{G} -GALOIS cover $\phi_X : X \rightarrow \overline{U}_\Upsilon$ we obtain $\phi'_X := \phi_X \circ \phi_{\overline{U}_\Upsilon} : X \rightarrow \mathbb{P}_x^1$. This morphism is étale at the points of the set $\{r_1, \dots, r_n\} := \phi_X^{-1}(\{x = 0\})$.

4.4.4. *Patching and deformation.* Now we must patch together and deform the G' -GALOIS cover $\phi_{W_X} : W_X \rightarrow X$ and the H' -GALOIS cover $\phi_{W_Y} : W_Y \rightarrow Y$. Let R , resp. A , be rings such that $\text{Spec}(R) = X \setminus \phi_X^{-1}(\{x = \infty\})$, resp. $\text{Spec}(A) = Y \setminus \phi_Y^{-1}(\{y = \infty\})$. Hence, $k[x] \subset R$ and $k[y] \subset A$. Let $F := (R \otimes_k A \otimes_k k[[t]])/(t - xy)$ and $T^f := \text{Spec}(F)$. Let T be the ZARISKI closure of T^f in $Z := X \times_{\text{Spec}(k)} Y \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$.

Observe that under the first, resp. second, projection of T inside Z , T is an X -scheme, resp. an Y -scheme. Let W_{XT} , resp. W_{YT} be the normalization of an irreducible dominating component of $W_X \times_X T$, resp. $W_Y \times_Y T$.

We next state the items of [Ku06, Proposition 6.3] which we will need and give a sketch of their proof. For a more detailed description see *loc. cit.*. We start with the following notation.

Notation. Let \mathcal{X} be a scheme over $\text{Spec}(k[[t]])$. We denote by $\tilde{\mathcal{X}}$ its completion along its closed fiber, *i.e.*, its (t) -adic completion. Let G_1 be a finite group and G_2 a subgroup of G_1 . Given a G_2 -GALOIS cover $Z' \rightarrow Z$ of smooth connected curves defined over k , it induces a disconnected G_1 -GALOIS cover $\text{Ind}_{G_2}^{G_1}(Z') \rightarrow Z$. The upper curve consists of as many copies of Z' as the index of G_2 in G_1 . For any scheme X and finite group \mathcal{H} , let $\mathcal{HN}(X)$ be the category of generically separable coherent locally free sheaves of \mathcal{O}_X -algebras S endowed with a \mathcal{H} -action which is transitive on the geometric generic fibers of $\text{Spec}(S) \rightarrow X$.

Further notation. Let $L := \text{Spec}(k[z])$ be an affine line. The k -algebra homomorphism $k[[t]][z] \rightarrow F$ defined by $z \mapsto x + y$ induces a $\text{Spec}(k[[t]])$ -morphism $\phi : T^f \rightarrow L^* := L \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$. Let $\lambda \in L$ be the closed point corresponding to the point $z = 0$. Since L is included in L^* as the closed fiber, λ can be viewed as a closed point of L^* corresponding to the maximal idea (z, t) in $k[[t]][z]$. Denote by $\{\tau_1, \dots, \tau_n\}$ the inverse image $\phi^{-1}(\lambda)$. The closed fiber of T is a reducible curve consisting of X and n copies of Y , and each of these copies of Y intersects X at one of the points $\{\tau_1, \dots, \tau_n\}$. (This is because the

locus of $t = 0$ is the same as that of $xy = 0$ in T , and the locus of $t = 0$ and $x + y = 0$ in T is the same as that of $x = 0$ and $y = 0$.) Thus, we can identify the points $\{\tau_1, \dots, \tau_n\}$ with the points $\{r_1, \dots, r_n\}$ on the X -part of the closed fibre. Notice that on the “ i -th copy” of Y , the point τ_i corresponds to $s \in Y$, the unique point lying over $y = 0$. Therefore, we actually consider $\{\tau_1, \dots, \tau_n\}$ to be points lying in T . Let $T^0 := T \setminus \{\tau_1, \dots, \tau_n\}$, $T_X := T^0 \setminus \{x = 0\}$ and $T_Y := T^0 \setminus \{y = 0\}$. For each $i = 1, \dots, n$, let $\hat{T}_i := \text{Spec}(\hat{\mathcal{O}}_{T, \tau_i})$. For each $i = 1, \dots, n$, let $\hat{K}_{X, r_i} := \text{frac}(\hat{\mathcal{O}}_{X, r_i})$, $\mathcal{K}_{X, r_i} := \text{Spec}(\hat{K}_{X, r_i}[[t]] \times_{\text{Spec}(k[y])} \mathcal{O}_{Y, s})$, $\hat{K}_{Y, s} := \text{frac}(\hat{\mathcal{O}}_{Y, s})$ and $\mathcal{K}_Y^i := \text{Spec}(\hat{K}_{Y, s}[[t]] \otimes_{\text{Spec}(k[x])} \mathcal{O}_{X, r_i})$.

One of the ingredients of the proposition is the following lemma proved in [Ku06, Lemma 6.2] which is analogous to [Ha94, Corollary 2.2].

Lemma 4.5. *For any finite group \mathcal{H} , the base change functor*

$$\mathcal{HN}(T) \rightarrow \mathcal{HN}(\widetilde{T}_X \cup \widetilde{T}_Y) \times_{\mathcal{HN}(\bigcup_{i=1}^n \mathcal{K}_{X, r_i} \cup \mathcal{K}_Y^i)} \mathcal{HN}\left(\bigcup_{i=1}^n \hat{T}_i\right)$$

is an equivalence of categories.

Proposition 4.6. [Ku06, Proposition 6.3] *Let $\phi_{W_X} : W_X \rightarrow X$ be an irreducible normal G' -cover which is étale over r_1, \dots, r_n . Similarly, take $\phi_{W_Y} : W_Y \rightarrow Y$ to be an irreducible normal H' -cover which is étale over s . Let W_{XT} , resp. W_{YT} , be the normalization of an irreducible dominating component of $W_X \times_X T$, resp. $W_Y \times_Y T$. Then there exists an irreducible normal Γ' -cover $W \rightarrow T$ satisfying the following conditions*

- (1) $W \times_T \widetilde{T}_X = \text{Ind}_{G'}^{\Gamma'}(W_{XT} \times_T \widetilde{T}_X)$.
- (2) $W \times_T \widetilde{T}_Y = \text{Ind}_{H'}^{\Gamma'}(W_{YT} \times_T \widetilde{T}_Y)$.
- (3) $W/H' \cong W_{XT}$ as an étale H' -cover of T .

Sketch of proof. Denote the right hand sides of (1) and (2) in the proposition by \widetilde{W}_X and \widetilde{W}_Y , respectively. The structural sheaf of their union \widetilde{W}^0 is an object of $\Gamma'\mathcal{N}(\widetilde{T}_X \cup \widetilde{T}_Y)$. Using the fact that ϕ_{W_X} is étale over each r_i and ϕ_{W_Y} is étale over s , one concludes that the restriction of \widetilde{W}^0 to $\bigcup_{i=1}^n (\mathcal{K}_{X, r_i} \cap \mathcal{K}_Y^i)$ is a Γ' -cover induced by the trivial cover. Denote by \hat{W}_i the Γ' -cover of \hat{T}_i induced by the trivial cover. The structural sheaf of $\hat{W} := \bigcup_{i=1}^n \hat{W}_i$ is an object of $\Gamma'\mathcal{N}(\bigcup_{i=1}^n \hat{T}_i)$. Moreover, the restriction of \hat{W} to $\bigcup_{i=1}^n (\mathcal{K}_{X, r_i} \cup \mathcal{K}_Y^i)$ is a Γ' -cover induced by the trivial cover. After having fixed an isomorphism between two of these induced covers one can apply Lemma 4.5. Then (1) and (2) follow immediately from this result. Finally to obtain (3), it suffices to note that both W/H' and W_{XT} restrict to the same G' -covers over the patches \widetilde{T}_X , \widetilde{T}_Y and \hat{T}_i , and then apply Lemma 4.5 again. \square

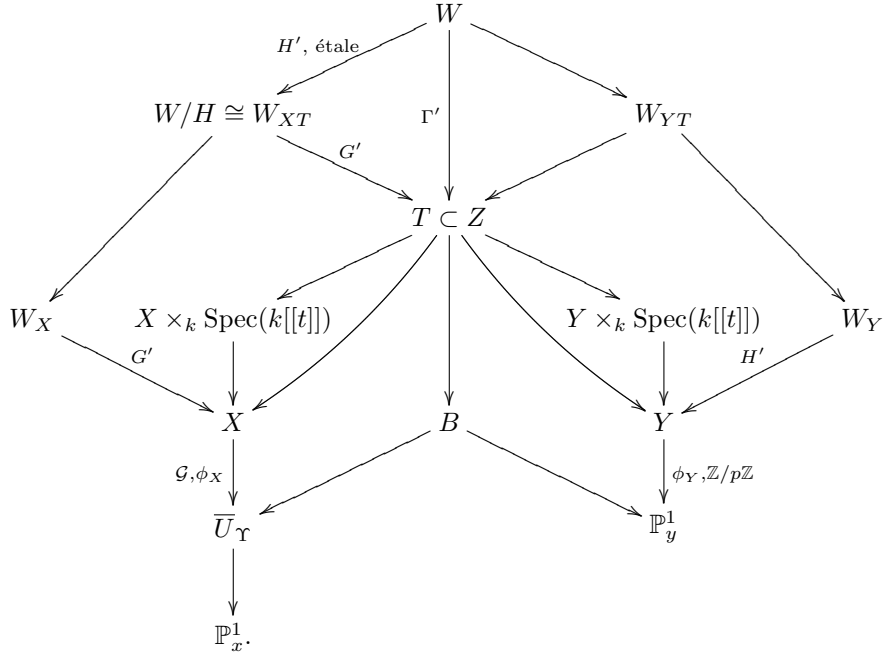
Taking $\phi_{W_X} : W_X \rightarrow X$ and $\phi_{W_Y} : W_Y \rightarrow Y$ to be as defined in Subsections 4.4.1-4.4.3, and using Proposition 4.6, we obtain an irreducible normal Γ' -cover $W \rightarrow T$ satisfying the conditions above. It remains to specialize this situation so that (as stated in the introduction to Section 4) we will have produced a finite étale GALOIS p -cover $T' \rightarrow U_T$ such that $\text{Gal}(k(T')/k(U_T))$ is a finite quotient of M_p and that the embedding problem in (3.4.1) defined by the pair of maps (α', φ'_1) admits a proper solution.

4.6.1. *The generic fiber.* For any $k[[t]]$ -scheme \mathcal{V} , let \mathcal{V}^g be its generic fiber over $k((t)) := \text{frac}(k[[t]])$. Denote by B the locus of $xy = t$ in $\overline{U}_\Gamma \times_{\text{Spec}(k)} \mathbb{P}_y^1 \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$. Let $\phi_X : X \rightarrow \overline{U}_\Gamma$, resp. $\phi_Y : Y \rightarrow \mathbb{P}_y^1$, be the cover of Subsubsection 4.4.1, resp. 4.4.2. The restriction of the map

$$\phi_X \times \phi_Y \times \text{id} : X \times_{\text{Spec}(k)} Y \times_{\text{Spec}(k)} \text{Spec}(k[[t]]) \longrightarrow \overline{U}_\Gamma \times_{\text{Spec}(k)} \mathbb{P}_y^1 \times_{\text{Spec}(k)} \text{Spec}(k[[t]])$$

to T gives a morphism $T \rightarrow B$.

The following diagram summarizes the situation (4.6.1)



Lemma 4.7. (cf. [Ku06, Lemma 6.4]) *Let X , Y and T be as in (4.6.1). Denote $\mathcal{G}' := \mathcal{G} \times \mathbb{Z}/p\mathbb{Z}$. The morphism $T \rightarrow B$ yields a \mathcal{G}' -GALOIS cover $T^g \rightarrow \overline{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))$.*

Proof. We observe first that $B^g = \overline{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))$. Next, it follows from the definition of T that the function field of its generic fiber T^g over $k((t))$ is the compositum of $L_1 := k(X) \otimes_k k((t))$ and $L_2 := k(Y) \otimes_k k((t))$. But these fields are linearly disjoint over $k((t))$. In fact, $k(X)$ is a finite extension of $k(x)$ and $k(Y)$ is a finite extension of $k(t/x)$ (recall that $y = t/x$). Hence $(k(X) \otimes_k k((t))) \cap (k(Y) \otimes_k k((t))) = k((t))$. The lemma now is a consequence of [FrJa05, Chapter 2, Lemma 2.5.6]. \square

Recall that $\mathcal{U}_\Gamma = \overline{U}_\Gamma \setminus U_\Gamma$. Since the branch locus of $W^g \rightarrow T^g$ is determined by that of $W \rightarrow T$, then $W^g \rightarrow T^g$ is ramified only at the points of T^g lying above those in $\mathcal{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))$. Moreover, $T^g \rightarrow \overline{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))$ is ramified only at the points of $\mathcal{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))$. So, we get a sequence of coverings with the following groups and ramification behavior :

$$(4.7.1) \quad \begin{array}{ccc} W^g & \xlongequal{\quad} & W^g \\ & \searrow^{\Gamma'} & \downarrow^{H', \text{ étale}} \\ & & W^g/H \cong W_{XT}^g \\ & & \downarrow^{G', \text{ branched only at points lying over } \mathcal{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))} \\ & & T^g \\ & & \downarrow^{\mathcal{G}', \text{ branched only at } \mathcal{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t)))} \\ & & \overline{U}_\Gamma \times_{\text{Spec}(k)} \text{Spec}(k((t))). \end{array}$$

4.7.1. *Specialization.* In order to obtain a similar sequences of coverings over the field k with the same ramification behavior we need the following specialization result.

Proposition 4.8. [Ku06, Proposition 6.6] *Let k be an algebraically closed field and X_0, \dots, X_3 normal projective curves defined over $\text{Spec}(k[[t]])$. Denote by X_i^g the corresponding generic fibers. Assume the following conditions are satisfied:*

- (1) *For $i = 1, 2, 3$, the curve X_i has generically smooth closed fiber denoted by X_i^0 .*
- (2) *For $i = 1, 2, 3$, there exists proper surjective morphisms $\psi_i : X_i \rightarrow X_{i-1}$ defined over $\text{Spec}(k[[t]])$. Denote by $\psi_i^g : X_i^g \rightarrow X_{i-1}^g$ the induced morphisms on the generic fibers.*
- (3) *The morphisms ψ_i from part (2) are GALOIS covers such that the group corresponding to ψ_1 is \mathcal{G}' , to ψ_2 is G' and to ψ_3 is H' . The composite morphism $\psi_3 \circ \psi_2$ is also a GALOIS cover with group Γ'*
- (4) *There exists a smooth projective curve Y_0 defined over k such that $X_0^g = Y_0 \times_{\text{Spec}(k)} \text{Spec}(k((t)))$.*
- (5) *The curve X_1^g has genus at least 1.*
- (6) *Let $\zeta_1, \dots, \zeta_r \in Y_0$ and $\xi_1, \dots, \xi_r \in X_0^g$ be the points induced by them. Suppose that the composite morphism $\psi_3^g \circ \psi_2^g \circ \psi_1^g$ is étale away from $\{\xi_1, \dots, \xi_r\}$.*

Then there exist smooth connected projective curves Y_1, Y_2, Y_3 defined over k and morphisms $\eta_i : Y_i \rightarrow Y_{i-1}$ for $i = 1, 2, 3$ such that their composite $\eta_3 \circ \eta_2 \circ \eta_1$ is étale away from $\{\zeta_1, \dots, \zeta_r\}$. Finally the morphisms $\eta_1, \eta_2, \eta_3, \eta_3 \circ \eta_2$ are GALOIS covers with groups \mathcal{G}' , G' , H' and Γ' , respectively.

Sketch of proof. The first observation is that since H' , G' and Γ' are finite groups, then the covers ψ_i actually descend to GALOIS covers of connected $\text{Spec}(\mathcal{A})$ -schemes $\psi_{i,\mathcal{A}} : X_{i,\mathcal{A}} \rightarrow X_{i-1,\mathcal{A}}$, where $\mathcal{A} \subset k[[t]]$ is a regular finite type $k[t]$ -algebra. Moreover, each $\psi_{i,\mathcal{A}}$ still has the same group as ψ_i . Let $E := \text{Spec}(\mathcal{A}[t^{-1}])$. By taking the base changes $X_{i,E} := X_{i,\mathcal{A}} \times_{\text{Spec}(\mathcal{A})} \text{Spec}(E)$ and considering the induced morphisms $\psi_{i,E}$, they still have the same groups. Note that $X_{0,E} \cong Y_0 \times_{\text{Spec}(k)} E$. Furthermore, for each $i = 1, \dots, r$, denote by $\zeta_{i,E}$ the point induced by ζ_i in $X_{0,E}$ for each $i = 1, \dots, r$ and observe that $\psi_{3,E} \circ \psi_{2,E} \circ \psi_{1,E}$ is only ramified over

$\{\zeta_{1,E}, \dots, \zeta_{r,E}\}$. The proof is completed by showing that there exists a non-empty open subset E' of E so that the fiber of $\psi_{i,E}$ over each closed point of E' is irreducible and non-empty. \square

It follows from Proposition 4.8 that there exist smooth connected projective curves W^s , W_{XT}^s and T^s defined over k such that we have a sequence of coverings with the same group and ramification behavior as (4.7.1)

$$(4.8.1) \quad \begin{array}{ccc} W^s & \xlongequal{\quad} & W^s \\ & \searrow \phi_{W^s, \Gamma'} & \downarrow H', \text{ étale} \\ & & W^s/H' \cong W_{XT}^s \\ & & \downarrow G', \text{ branched only at points lying over } \mathcal{U}_\Upsilon \\ T' := \phi_{T^s}^{-1}(\mathcal{U}_\Upsilon) & \xrightarrow{\quad} & T^s \\ & & \downarrow \phi_{T^s, \mathcal{G}'}, \text{ branched only at } \mathcal{U}_\Upsilon \\ & & \overline{U}_\Upsilon. \end{array}$$

The Γ' -GALOIS cover ϕ_{W^s} (which is étale over T') gives a proper solution $\lambda'_{T'}$ of the embedding problem

$$(4.8.2) \quad \begin{array}{ccc} & \pi_1(T') & \\ \lambda'_{T'} \swarrow & \downarrow \varphi'_{T'} & \\ \Gamma' & \xrightarrow{\alpha} & G'. \end{array}$$

Remark 4.9. Let \mathcal{G}' be as in Lemma 4.7. It follows from Lemma 2.8 and Remark 3.3 that \mathcal{G}' is indeed a quotient of M_p .

4.10. Solving (2.10.1) for N .

As T' is a curve, $\pi_1(T')$ is projective and so by Lemma 2.11 we have a proper solution $\lambda_{T'}$ of the following embedding problem :

$$(4.10.1) \quad \begin{array}{ccc} & \pi_1(T') & \\ \lambda_{T'} \swarrow & \downarrow \varphi_{T'} & \\ \Gamma & \xrightarrow{\alpha} & G. \end{array}$$

Thus we are now in the situation described by the following diagram

$$(4.10.2) \quad \begin{array}{ccccc} N & \xrightarrow{\quad} & \pi_1(T') & \xrightarrow{\quad} & \pi_1(C) \\ \psi \downarrow & \varphi_{T'} \swarrow & \downarrow \varphi & \searrow & \\ \Gamma & \xrightarrow{\alpha} & G, & & \end{array}$$

where $\text{Ker}(\alpha)$ is minimal normal and is contained in $M(\Gamma)$. Denote $\Psi := (\lambda_{T'})_{1N}$. Observe that $\alpha(\lambda_{T'}(N)) = \varphi_{T'}(N) = \psi(N) = G$. Whence $\lambda_{T'}(N) \cdot M(\Gamma) = \Gamma$, because $H \leq M(\Gamma)$. Since $\lambda_{T'}(N)$ is a normal subgroup of Γ , we conclude from [RiZa00, Proposition 8.3.6] that $\Psi(N) = \Gamma$, *i.e.*, Ψ is a proper solution to the embedding problem (2.10.1).

4.11. **The case where p divides the order of H .** Since H is minimal normal in Γ , then H is generated by its Sylow- p subgroups, i.e., H is a *finite quasi- p group*. It follows from [Po95, Theorem B] or [Ha03, Corollary 4.6] that every finite quasi- p embedding problem of smooth connected affine curves defined over an algebraically closed field of characteristic p has a proper solution. Now, it suffices to repeat the argument of Subsection 4.10. \square

5. PROOF OF THEOREM 1.1 AND GROUP THEORETIC CONSEQUENCES

In this section we obtain group theoretic consequences from Theorems 2.7 and 2.9. In particular we prove in Corollary 5.5 that given a normal subgroup N of $\pi_1(C)$, every proper open subgroup of N is a free profinite group. As a consequence, we obtain Kumar's result [Ku06, Theorem 4.1] showing that the commutator subgroup of $\pi_1(C)$ is a free profinite group of countable rank (see Corollary 5.7).

Definition 5.1. A profinite group is said to be *virtually free*, if it has an open free subgroup.

Theorem 5.2. *Let C , resp. C_1 , resp. C_2 , be smooth connected affine curves defined over an algebraically closed field of characteristic $p > 0$ of countable cardinality. Let $\pi_1(C)$, resp. $\pi_1(C_1)$, resp. $\pi_1(C_2)$, be their respective algebraic fundamental groups. Let N be a normal subgroup of $\pi_1(C)$ such that a Sylow- p subgroup M_p of the quotient group $M = \pi_1(C)/N$ is infinitely generated. Then*

- (1) N is a virtually free profinite group.
- (2) Let $G_1 \triangleleft \pi_1(C_1)$ and $G_2 \triangleleft \pi_1(C_2)$ be two normal subgroups such that the quotients $\pi_1(C_1)/G_1$ and $\pi_1(C_2)/G_2$ have infinitely generated Sylow- p subgroups. Then G_1 and G_2 are isomorphic if and only if $G_1/M_S(G_1) \cong G_2/M_S(G_2)$ for every finite simple group S .
- (3) N is a free profinite group of countable rank if and only if $N/M_S(N)$ is infinite for every finite simple group S .

Proof. Theorem 2.9 together with [RiZa00, Corollary 8.5.8] implies (1) and together with [RiZa00, Theorem 8.5.2] implies (2). Item (3) is a consequence of (2) and Theorem 2.7 (item (4)). \square

The proof of Theorem 1.1 follows from Theorem 2.9 and the next two theorems which are proved similarly as in [Za05, Theorem 2.4, Corollary 2.5 and Theorem 2.6], respectively. For the convenience of the reader we include their proofs, taking into account that algebraic fundamental groups under consideration here (in contrast with the previous paper) are those of smooth connected affine curves over algebraically closed fields of positive characteristic and countable cardinality.

Theorem 5.3. *Let $\pi_1(C)$ be the algebraic fundamental group of a smooth connected affine curve C defined over an algebraically closed field k of characteristic $p > 0$ of countable cardinality. Let N be a normal subgroup of $\pi_1(C)$ such that a Sylow- p subgroup M_p of the quotient group $M = \pi_1(C)/N$ is infinitely generated. Then N is isomorphic to a normal subgroup of a free profinite group of countable rank.*

Proof. By Theorems 2.9 and 2.7 (item (1)), N is isomorphic to an accessible subgroup of infinite index of a free profinite group of countable rank. Therefore, by Theorem 2.7 (item (2)), we just need to show that $N/M_l(N)$ is infinite or trivial for

any prime l . Note that by definition $N/M_l(N)$ is the maximal elementary abelian pro- l quotient of N

Suppose $N/M_l(N)$ is finite. Then $M_l(N)$ is open in N and thus there exists an open subgroup \mathcal{M} of $\pi_1(C)$ such that $M_l(\mathcal{M}) \cap N = M_l(N)$. The group \mathcal{M} is isomorphic to the algebraic fundamental group $\pi_1(U)$ of a finite étale cover U of C . Moreover, N is the intersection of such \mathcal{M} 's. Put $\mathcal{M}^{(l)} := \mathcal{M}/R_l(\mathcal{M})$, $\overline{N} := NR_l(\mathcal{M})/R_l(\mathcal{M})$.

If $l = p$, then $\mathcal{M}^{(p)}$ is a free pro- p group of countable rank (cf. [Ha00, Corollary 3.7]). Moreover, as \overline{N} is a normal subgroup of $\mathcal{M}^{(p)}$, it is a free pro- p group of countable rank, unless it is trivial (cf. [RiZa00, Proposition 8.6.3]). Hence, $N/M_p(N)$ is infinite or trivial, as needed.

Suppose $l \neq p$. Observe now that we can choose \mathcal{M} satisfying the property $M_l(\mathcal{M}) \cap N = M_l(N)$ with index in $\pi_1(C)$ as large as we wish. In particular, the genus g of the smooth compactification \overline{U} of U and the finite number r of points of $\overline{U} \setminus U$, may be taken arbitrarily large. Let $\pi_1(U) \twoheadrightarrow \pi_1(U)^{(l)}$ be the quotient homomorphism to the maximal pro- l quotient of $\pi_1(U)$. It is known from the last statement of [SGA1, Exp. XIII, Corollaire 2.12] that the maximal pro-prime-to- p quotient $\pi_1(U)^{(p')}$ of $\pi_1(U)$ is a free profinite group on $t = 2g + r - 1$ generators. Taking its maximal pro- l -quotient it coincides with $\pi_1(U)^{(l)}$. Hence the latter group is a free pro- l of rank $t = 2g + r - 1$. We deduce from all this reasoning an epimorphism $\Upsilon \twoheadrightarrow \hat{F}_t$, where the latter group is a free pro- l group of rank t . This epimorphism factors through $\mathcal{M}^{(l)}$ and since \overline{N} is a normal subgroup of $\mathcal{M}^{(l)}$ it has rank at least t . But t can be chosen larger than the minimal number of generators of $N/M_l(N)$, this yields a contradiction. \square

Remark 5.4. In fact [Ha00, Corollary 3.7] proves a bit more than what was stated above. Although we do not require it for our purposes, it follows from the above that for every smooth affine connected variety X defined over an algebraically closed field k of characteristic $p > 0$, the maximal pro- p quotient $\pi_1(X)^{(p)}$ of the algebraic fundamental group $\pi_1(X)$ of X is free in as many generators as the cardinality of k .

In the case of smooth connected projective curves X , it was already known to SHAFAREVIČ [Sh47, Theorem 2] that $\pi_1(X)$ is free in γ_X generators, where γ_X denotes the p -rank of X (i.e., the \mathbb{F}_p -dimension of the p -torsion subgroup $J_X[p]$ of the Jacobian variety J_X of X). The number γ_X is bounded above by the genus of X . For a proof of the same result in the spirit of étale cohomology cf. [Cr84, Theorem 1.9].

Corollary 5.5. *Every proper open subgroup of N is a free profinite group.*

Proof. The result follows from the preceding theorem combined with [RiZa00, Theorem 8.7.1]. \square

Theorem 5.6. *Let $\pi_1(C)$ be the algebraic fundamental group of a smooth connected affine curve C defined over an algebraically closed field k of characteristic $p > 0$ of countable cardinality. Let N be a characteristic subgroup of $\pi_1(C)$ such that there exists a Sylow- p subgroup M_p of $M = \pi_1(C)/N$ which is infinitely generated. Then N is isomorphic to a characteristic subgroup of a free profinite group of countable rank.*

Proof. Let S be a finite simple group. By Theorem 2.7 (item (3)), we just need to show that $N/M_S(N)$ is either infinite or trivial. If S has a prime order l , then this is already contained in the proof of Theorem 5.3.

Suppose now that $N/M_S(N)$ is finite non-trivial. Then there exists a proper open normal subgroup \mathcal{M} of $\pi_1(C)$ containing N such that $M_S(\mathcal{M}) \cap N = M_S(N)$. Note that \mathcal{M} is isomorphic to $\pi_1(U)$ for some finite étale GALOIS cover U of C .

Assume S is a non-abelian simple group of order divisible by p and let Γ be a finite direct product of S . By the simplicity of S and the divisibility by p condition, S is quasi- p , i.e., is generated by its Sylow p -subgroups. Next consider the embedding problem $(\pi_1(U) \twoheadrightarrow \{1\}, \alpha : \Gamma \twoheadrightarrow \{1\})$. Since $\text{Ker}(\alpha) = \Gamma$ is a quasi- p group, then by [Po95, Theorem B] or [Ha03, Corollary 4.6], the latter embedding problem admits a proper solution $\pi_1(U) \twoheadrightarrow \Gamma$. Therefore we also have an epimorphism $\mathcal{M}/M_S(\mathcal{M}) \twoheadrightarrow \Gamma$. *A fortiori*, $\mathcal{M}/M_S(\mathcal{M})$ cannot be finite. Hence, $\mathcal{M}/M_S(\mathcal{M})$ equals an infinite direct product of copies of S , whence it does not have finite characteristic subgroups, which yields a contradiction. Consequently, $\#(N/M_S(N))$ is prime to p .

Observe now that we can choose \mathcal{M} satisfying the property $M_S(\mathcal{M}) \cap N = M_S(N)$ with index in $\pi_1(C)$ as large as we wish. In particular, the genus g of the smooth compactification \bar{U} of U and the finite number r of points of $\bar{U} \setminus U$, may be taken arbitrarily large. Again we consider the quotient map $\pi_1(U) \twoheadrightarrow \pi_1(U)^{(p')}$ to the maximal pro-prime-to- p quotient of $\pi_1(U)$. It follows from the last statement of [SGA1, exp. XIII, corollaire 2.12] that $\pi_1(U)^{(p')}$ is isomorphic to the maximal pro-prime-to- p quotient of a free profinite group of rank $2g + r - 1$, so it is free of rank $t = 2g + r - 1$. We deduce from all this reasoning an epimorphism $\mathcal{M} \twoheadrightarrow \hat{F}_t$, where the latter group is a free pro-prime-to- p group of rank t . Thus, $\hat{F}_t/M_S(\hat{F}_t)$ is a quotient of $\mathcal{M}/M_S(\mathcal{M})$. Since the order of $\hat{F}_t/M_S(\hat{F}_t)$ tends to infinity as t does, we may assume that the order of $\Upsilon/M_S(\mathcal{M})$ is larger than the order of $N/M_S(N)$. But $N/M_S(N)$ is characteristic in $\mathcal{M}/M_S(\mathcal{M}) \cong \prod S$, so $N/M_S(N) = \{1\}$, because a finite product $\prod S$ does not have any proper characteristic subgroup. \square

As a corollary we obtain the main result of Kumar [Ku06, Theorem 4.1].

Corollary 5.7. *The commutator subgroup of $\pi_1(C)$ is a free profinite group of countable rank.*

Proof. Let K be the commutator subgroup of $\pi_1(C)$. Since this group is characteristic and $\pi_1(C)/K$ has an infinitely generated Sylow- p subgroup (cf. [Ha00, Corollary 3.7]), by the preceding theorem it is isomorphic to a characteristic subgroup of a free profinite group of countable rank. It then follows from Theorem 2.7 (item (3)) that $K/M_S(K)$ is either trivial or infinite. Hence, by Theorem 5.2 (item (3)), it suffices to show that $K/M_S(K)$ is non-trivial for every finite simple group S . If S is non-abelian, then $\pi_1(C)/M_S(\pi_1(C))$ is a nontrivial quotient of K , so $K/M_S(K)$ is non-trivial. For S of prime order $l \neq p$, we observe that the maximal pro- l quotient $\pi_1(C)^{(l)}$ is isomorphic to the maximal pro- l -quotient of a free profinite group of rank $2g + r - 1$, so it is free pro- l of rank $t = 2g + r - 1$. So taking an open subgroup \mathcal{M} of $\pi_1(C)$ containing K of sufficiently large index we see that its maximal pro- l quotient $\mathcal{M}^{(l)}$ is not abelian, and therefore the image of K there is non-trivial (since K contains the commutator of \mathcal{M}). Similarly, when $l = p$, it follows from Remark 5.4 that $\pi_1(C)^{(p)}$ is a free pro- p group with rank \aleph_0 . In particular, by choosing an open subgroup \mathcal{M} of $\pi_1(C)$ we arrive at the same conclusion (i.e. that the image of K in $\mathcal{M}^{(p)}$ cannot be trivial). \square

Remark 5.8. In the final remarks of his thesis (*cf.* [Ku07, §7.3 and Theorem 7.13]) KUMAR briefly observed the following fact: the proof of his main theorem [Ku07, Theorem 5.8] also works if the commutator subgroup in the embedding problem in [Ku07, Proposition 5.7] is replaced by the kernel of an epimorphism of $\pi_1(C)$ to an infinitely generated abelian pro- p group. This result is also covered by our Theorem 2.9. Next we give an example that is covered by our theorem, but not covered by the results of KUMAR.

Example 5.9. Let p be a prime number and let S be a simple non-abelian group containing a non-trivial SYLOW p -subgroup. Let $M = (\prod S) \times C_p$ be a direct product of a countable direct power of S by a group of order p . Then M is quasi- p group (*i.e.*, it is generated by its SYLOW p -subgroups). Moreover, $M = \varprojlim_J M_J$, where J ranges over finite sets and $M_J = \prod_J S$ a finite direct power of S . Since for $|J_1| > |J_2|$ the embedding problem

$$(5.9.1) \quad \begin{array}{ccc} & & \pi_1(C) \\ & & \downarrow \psi \\ M_{J_1} & \xrightarrow{\alpha} & M_{J_2}. \end{array}$$

admits a proper solution (because the kernel of α is a direct product of a finite number of copies of S , and so it is a finite quasi p -group, *cf.* [Po95, Theorem B]), the universal property of the inverse limit provides an epimorphism $\eta_1 : \pi_1(C) \twoheadrightarrow M$. Thus we can write M as a quotient $M = \pi_1(C)/N$ of $\pi_1(C)$ by a normal subgroup N . The quotient group M/C_p has an infinitely generated Sylow p -subgroup. The composition of the epimorphism η_1 with the natural quotient epimorphism $\eta_2 : M \twoheadrightarrow M/C_p$, induces an epimorphism $\eta_3 : \pi_1(C) \twoheadrightarrow M/C_p$. Therefore, M/C_p can also be written as a quotient $\pi_1(C)/N_p$ of $\pi_1(C)$ by a normal subgroup N_p . It now follows from Theorem 2.9 that N_p is homogeneous. Since N is open in N_p , we conclude that it is indeed free (as actually any proper open subgroup of N_p) by Corollary 5.5. \square

6. APPENDIX

We rewrite here the proof of Theorem 2.2 in [Za05], where the embedding problem (2.6.1) is properly solved for a projective normal subgroup N of the algebraic fundamental group Γ of a smooth connected projective curve of genus $g > 1$ defined over an algebraically closed field of characteristic zero. Recall that this group is just the profinite completion of the topological fundamental group Π^{top} of a compact RIEMANN surface of genus $g > 1$ (*cf.* [SGA1, Exposé X, end of p. 208 and beginning of p. 209]). The group Π^{top} admits a presentation of the form $\Pi^{\text{top}} = \langle x_i, y_i \mid \prod_{i=1}^g [x_i, y_i] = 1, i = 1, \dots, g \rangle$. Here we exploit this presentation; in particular we rely heavily on the relation $\prod_{i=1}^g [x_i, y_i] = 1$. In the case of positive characteristic an explicit presentation for $\pi_1(C)$ is not available. Therefore, in order to get (as we did) a similar statement in positive characteristic (for smooth connected affine curves), it was necessary to use methods from algebraic geometry which allow us to construct solutions for embedding problems.

The present proof simplifies the proof of [Za05] and eliminates small but embarrassing errors. Note also that we do not need here splitting of the embedding problem.

Lemma 6.1. *Let $\Gamma = \langle x_i, y_i \mid \prod_{i=1}^g [x_i, y_i] \rangle$ be the algebraic fundamental group of a smooth connected projective curve of genus $g > 1$ defined over an algebraically closed field of characteristic zero. Consider the following finite embedding problem:*

$$(6.1.1) \quad \begin{array}{ccc} & & \Gamma \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & B. \end{array}$$

Suppose it admits a weak solution $\varphi : \Gamma \rightarrow A$ such that $\varphi(x_1) = \varphi(x_2) = \dots = \varphi(x_{sn+s})$ and $\varphi(y_1) = \varphi(y_2) = \dots = \varphi(y_{sn+s})$, where $n = |K\varphi(\Gamma)|$ and s is the minimal number of generators of $K = \ker(\alpha)$. Then (6.1.1) admits a proper solution.

Proof. We use the notation $[x, y]$ for $x^{-1}y^{-1}xy$ and \bar{g} for $\varphi(g)$ in the argument that follows. Choose a minimal set of generators k_1, \dots, k_s of K . Let η be a map that sends $x_{in+1}, x_{in+2}, \dots, x_{in+n}$ to $\bar{x}_{sn+i+1}k_{i+1}$ for all $i = 0, \dots, s-1$ and coincides with φ on the other generators. Then η extends to a homomorphism if

$$[\eta(x_1), \eta(y_1)] \dots [\eta(x_g), \eta(y_g)] = 1$$

(since this would mean that the homomorphism from a free profinite group $F(x_1, y_1, \dots, x_g, y_g) \rightarrow A$ extending η factors through Γ).

Since $[\bar{x}_{in+1}k_i, \bar{y}_{in+1}] \in K\varphi(\Gamma)$ one has $([\bar{x}_{in+1}k_i, \bar{y}_{in+1}])^n = 1 = ([\bar{x}_{in+1}, \bar{y}_{in+1}])^n$. Therefore $[\eta(x_1), \eta(y_1)] \dots [\eta(x_{sn}), \eta(y_{sn})] = \prod_{i=0}^{s-1} ([\bar{x}_{in+1}k_i, \bar{y}_{in+1}])^n = 1$ as well as $[\bar{x}_1, \bar{y}_1] \dots [\bar{x}_{sn}, \bar{y}_{sn}] = \prod_{i=0}^{s-1} ([\bar{x}_{in+1}, \bar{y}_{in+1}])^n = 1$. But

$$[\eta(x_{sn+1}), \eta(y_{sn+1})] \dots [\eta(x_g), \eta(y_g)] = [\bar{x}_{sn+1}, \bar{y}_{sn+1}] \dots [\bar{x}_g, \bar{y}_g].$$

It follows that $[\eta(x_1), \eta(y_1)] \dots [\eta(x_g), \eta(y_g)] = [\bar{x}_1, \bar{y}_1] \dots [\bar{x}_g, \bar{y}_g] = 1$ as needed.

Thus η extends to a homomorphism $\psi : \Gamma \rightarrow A$ such that $f = \alpha\psi$. But

$$\psi(x_1^{-1}x_{sn+1}) = k_1, \dots, \psi(x_{n(s-1)+1}^{-1}x_{ns+s}) = k_s$$

so ψ is an epimorphism and the lemma is proved. \square

The next lemma shows that by replacing Γ with an open subgroup of sufficiently large index, we can reduce ourselves to the situation of Lemma 6.1.

Lemma 6.2. *Let N be a projective subgroup of Γ and consider the embedding problem*

$$(6.2.1) \quad \begin{array}{ccc} & & N \\ & & \downarrow f \\ A & \xrightarrow{\alpha} & B, \end{array}$$

where A, B are finite. Then there exists an open subgroup U of Γ containing N and an embedding problem

$$(6.2.2) \quad \begin{array}{ccc} & & U \\ & & \downarrow \eta \\ A & \xrightarrow{\alpha} & B, \end{array}$$

satisfying hypothesis of Lemma 6.1 such that the restriction $\eta|_N = f$.

Proof. Since N is projective there exists a homomorphism $f' : N \rightarrow A$ such that $\alpha\varphi(N) = B$. Put $B' = f'(N)$.

By Lemma 8.3.8 in [RZ-2000] there exists an open subgroup U of Γ_g containing N and an epimorphism $\varphi : U \rightarrow B'$ such that $\varphi|_N = f'$. Since an open subgroup of Γ_g is again a profinite surface group, replacing Γ_g by U we may assume the existence of the following commutative diagram:

$$(6.2.3) \quad \begin{array}{ccc} & N & \longrightarrow \Gamma_g \\ & \searrow f' & \nearrow \varphi \\ B' & & \\ \downarrow & & \downarrow f \\ A & \xrightarrow{\alpha} & B, \end{array}$$

where the top horizontal map is the natural inclusion. Let U_i be the family of all open subgroups of Γ_g containing N . Then $\varphi_i := \varphi|_{U_i}$ is an epimorphism for every i . Note that every U_i is again a profinite surface group and so has a presentation $U_i = \langle x_1, y_1, \dots, x_{g_i}, y_{g_i} \mid \prod_{j=1}^{g_i} [x_j, y_j] \rangle$, where the genus g_i of U_i can be computed by the formula $g_i - 1 = [\Gamma_g : U_i](g - 1)$ (we abusing notation of course, for different i , the elements $x_1, y_1 \dots$ are not the same and not the same as those in Γ). This means that we can choose i with the number of generators of U_i sufficiently large, so that there exists i with $\varphi(x_{j_1}) = \dots = \varphi(x_{j_{sn+s}})$ and $\varphi(y_{j_1}) = \dots = \varphi(y_{j_{sn+s}})$, where $n = |K||B'|$, s is the minimal number of generators of K and $j_l > j_k$ whenever $l > k$. We shall use the notation x^y for $y^{-1}xy$ in the argument to follow. Suppose $j_1 \neq 1$. Then $\prod_{j=1}^{g_i} [x_j, y_j] = [x_{j_1}, y_{j_1}]([x_1, y_1] \cdots [x_{j_1-1}, y_{j_1-1}])^{[x_{j_1}, y_{j_1}]} [x_{j_1+1}, y_{j_1+1}] \cdots [x_{g_i}, y_{g_i}]$ so replacing the generators $x_1, y_1, \dots, x_{j_1-1}, y_{j_1-1}$ by $x_1^{[x_{j_1}, y_{j_1}]}, y_1^{[x_{j_1}, y_{j_1}]}, \dots, x_{j_1-1}^{[x_{j_1}, y_{j_1}]}, y_{j_1-1}^{[x_{j_1}, y_{j_1}]}$ we may assume that $j_1 = 1$. Continuing similarly, we in fact may assume that $\varphi(x_1) = \varphi(x_2) = \dots = \varphi(x_{sn+s})$ and $\varphi(y_1) = \varphi(y_2) = \dots = \varphi(y_{sn+s})$. \square

We now solve the embedding problem for an accessible projective subgroup N , having N normal as a particular case. Note that only in this proof we assume the minimality of the kernel K .

Theorem 6.3. *A projective accessible subgroup N of Γ is isomorphic to an accessible subgroup of a free profinite group.*

Proof. By Theorem 2.7 we just need to solve the following embedding problem for N :

$$(6.3.1) \quad \begin{array}{ccc} & N & \\ & \downarrow f & \\ A & \xrightarrow{\alpha} & B, \end{array}$$

where A, B are finite, $K := \text{Ker}(\alpha)$ is minimal normal and $K \leq M(A)$.

By two preceding lemmas we have an open subgroup $N \leq U \leq \Gamma$ and an epimorphism $\psi : U \rightarrow A$ such that $\alpha(\psi(N)) = B$ and so $\psi(N)M(A) = A$. Since $\psi(N)$ is a subnormal subgroup of A by Proposition 8.3.6 in [RZ] $\psi(N) = A$ as needed. \square

REFERENCES

- [Bo04] N. Borne, *A relative Shafarevich theorem*, Math. Zeitschrift **248** (2004), 351–367.
- [Ei95] D. Eisenbund, *Commutative algebra with a view towards algebraic geometry*, Springer-Verlag, 1995.
- [SGA1] A. Grothendieck, *Séminaires de géométrie algébrique 1*, Documents Mathématiques 3, Soc. Math. France, 2003.
- [Cr84] R. Crew, *Étale p -covers in characteristic p* , Compositio Math. **52** (1984), 31-45.
- [FrJa05] M. D. Fried, M. Jarden, *Field arithmetic*, Springer-Verlag, 2005, second revised edition.
- [Ha94] D. Harbater, *Abhyankar’s conjecture on Galois groups over curves*, Inventiones Math. **117** (1994), 1-25.
- [Ha00] D. Harbater, *Embedding problems with local conditions*, Israel J. Math. **118** (2000), 317-355.
- [Ha03] D. Harbater, *Abhyankar’s Conjecture and embedding problems*, J. reine und angew. Math. (Crelle) **559** (2003), 1-24.
- [HaSt05] D. Harbater, K. F. Stevenson, *Local Galois Theory in dimension two*, Adv. Math. **198** (2005), 623-653.
- [Ku06] M. Kumar, *Fundamental groups in positive characteristic*, preprint arxiv, 2006.
- [Ku07] M. Kumar, *Fundamental groups in positive characteristic*, PhD thesis, University of Purdue, 2007, <http://www.math.msu.edu/~kumar/Publication/thesis.pdf>.
- [Me78] O. V. Melnikov, *Normal subgroups of free profinite groups*, Math. USSR Izvestija **12** (1978), 3-25.
- [PaSt00] A. Pacheco, K. F. Stevenson, *Finite quotients of the algebraic fundamental group of projective curves in positive characteristic*, Pacific J. of Math. **192** (2000), no. 1, 143–158.
- [Po95] F. Pop, *Étale Galois covers of affine smooth curves*, Invent. Math. **120** (1995), 555–578.
- [Pr06] R. Pries, *Wildly ramified covers with large genus*, J. Number Th. **119** (2006), 194-209.
- [Ra94] M. Raynaud, *Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d’Abhyankar*, Inventiones Math. **116** (1994), 425-462.
- [RiZa00] L. Ribes, P. Zalesskii, *Profinite groups*, Springer-Verlag, 2000.
- [Sh47] I. Shafarevich, *On p -extensions*, Math USSR Mat. Sbornik **20** (1947), 351-363.
- [St96] K. F. Stevenson, *Galois groups of projective curves in characteristic p* , J. of Algebra. **182** (1996) 777-804.
- [Za05] P. Zalesskii, *Profinite surface groups and the congruence kernel of arithmetic lattices in $SL_2(\mathbb{R})$* , Israel J. Math. **146** (2005), 111-123.

UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, INSTITUTO DE MATEMÁTICA, RUA GUAIAQUIL
83, CACHAMBI, 20785-050 RIO DE JANEIRO, RJ, BRASIL
E-mail address: `amilcar@acd.ufrj.br`

CALIFORNIA STATE UNIVERSITY
E-mail address: `katherine.stevenson@csun.edu`

UNIVERSIDADE DE BRASÍLIA
E-mail address: `pz@mat.unb.br`